# Implied Impermanent Loss for Concentrated Liquidity

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# Yield Farming by Liquidity Provision

- Situation: Trade BTC for ETH on a Decentralized Exchange (DEX).

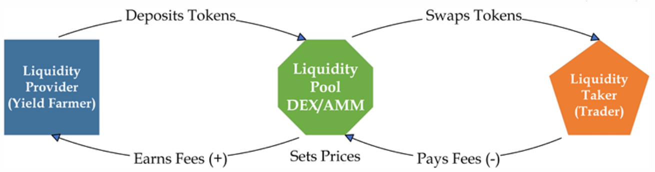



Figure: **Yield Farming via Liquidity Provision**

- On a DEX/"v2", AMM: $L = \sqrt{N_1 N_2}$, where $L$ is the total liquidity, and $N_1$ and $N_2$ are the amounts of tokens 1 and 2.
- If a trader wants 1 unit of token 2 from the pool, the AMM requires $\Delta_1$ units of token 1, where $\Delta_1$ satisfies $\sqrt{(N_1 + \Delta_1)(N_2 - 1)} = L$.
- Impermanent Loss (IL): if the fundamental value of the tokens changes, an arbitrageur trades at the stale price in the direction of the price change, thus minimizing the pool's value (adverse selection). details

# This Paper - Main Results

- Option-Implied Impermanent Loss (IIL)
    - Replication of IIL for 7 pools across maturities (7 to 30 days).
    - As expected, IIL for concentrated liquidity (i.e. "v3") is higher when liquidity is concentrated.
    - Higher IIL in more exotic pools (for example, ETH-BNB).
    - Introduced Impermanent Loss Risk Premia (IIL - RIL) for concentrated liquidity.
- LP's Risk and Risk Compensation
    - Higher IIL leads LPs to widen their liquidity ranges.
    - Higher ILRP reflects stronger compensation, leading LPs to re-center liquidity closer to the spot price.
- Mathematical Framework
    - Risk-neutral valuation of IIL, using a unified approach based on traded options.
    - Estimating the risk-neutral density as no options on the relative price are available.

# Literature Review

This paper bridges the literature on the risks of decentralized liquidity provision and the option-implied pricing information.

1. Literature on Liquidity Provision on DEXs and IL
   Li et al. (2024), Heimbach et al. (2022), Milionis et al. (2024), Fukasawa et al. (2023), Cartea et al. (2024), Lehar and Parlour (2023), Clark (2020), Clark (2021)
2. Literature on Option-Implied Information (mostly for equities):
   $IV/VRP$: Carr and Wu (2009), Bollerslev et al. (2009), Drechsler and Yaron (2011), Bollerslev et al. (2014)
   $IV_{dn}/IV_{up}$: Feunou et al. (2018), Kilic and Shaliastovich (2019)
   $IV/VRP$ for BTC: Alexander and Imeraj (2021)
3. $IIL$ $v2$ for BTC-ETH (30 days): Find that ILRP is state-dependent but unconditionally zero
   Papanicolaou et al. (2025)

# Impermanent Loss in Continuous Time ($IL^{v2}$)

- AMM with constant product rule and with (exogenous) prices (real-world dynamics) of tokens following

$$dP_i\left(t\right) = \mu_i P_i\left(t\right) dt + \sigma_i\left(t\right) P_i\left(t\right) dB_i\left(t\right), \quad i = 1,\, 2, \tag{1}$$

  where $B_i(t)$ for $i = 1$ and 2 are two correlated Brownian motions.

- IL is the integrated variance of the relative price ($R(t) = P_1(t)/P_2(t)$) of the two tokens (details, Papanicolaou et al. (2025), Milionis et al. (2024)),

$$dIL\left(t\right) = \frac{1}{8}\sigma_R^2\left(t\right) dt, \tag{2}$$

where

$$\sigma_R\left(t\right) = \sqrt{\sigma_1^2\left(t\right) - 2\rho\sigma_1\left(t\right)\sigma_2\left(t\right) + \sigma_2^2\left(t\right)}\,. \tag{3}$$

- IL is a function of the tokens' variances and their correlation.

# Implied Impermanent Loss ($IIL^{v2}$) – Method & Challenges

- Goal: price IL via a variance-swap representation on the relative price

$$R(t) = \frac{P_1(t)}{P_2(t)} \ .$$

- Key theoretical challenges:
  - $R(t)$ is not a $\mathbb{Q}$-martingale $\Rightarrow$ standard variance-swap pricing fails.
  - The log contract on $R(T)$ is not directly replicable since options trade only on individual tokens.
- Methodological solution (see Papanicolaou et al. (2025)):
  - Change of numéraire to $\widetilde{\mathbb{Q}}$ under which $R(T)$ is a martingale.
  - Construct a joint risk-neutral density for $(P_1(T), P_2(T))$ using options on individual tokens.
  - Use Hansen–Jagannathan bounds to identify a joint pricing kernel.
- We obtain a model-free implied measure of impermanent loss,

$$\widetilde{\mathbb{E}}^{\mathbb{Q}}[IL(T)] = \frac{1}{8T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\left[\int_0^T \sigma_R^2(t)\,dt\right] \ .$$

# Implied Impermanent Loss ($IIL^{v3}$) I

- In **concentrated liquidity**, LPs allocate capital within ranges (e.g., $\pm 20\%$ around the relative price).
- $IIL^{v3}$ can be computed by integrating the total IL over time, equivalent to a corridor variance swap (Lee (2010a) and Lee (2010b)),

$$\begin{aligned}
\widetilde{\mathbb{E}}^Q[IL_i(T)] =& \widetilde{\mathbb{E}}^Q \left[ \int_0^T \underbrace{\frac{1}{4} \frac{\sigma_R^2(t)\sqrt{R(t)}}{2\sqrt{R(t)} - \sqrt{r_i} - \frac{R(t)}{\sqrt{r_{i+1}}}} \mathbf{1}_{\{R(t)\in C_i\}} dt}_{dIL_i(t)} \right] \\
=& \frac{1}{2} \int_{K\in C_i} \frac{P(K)\mathbf{1}_{\{K<R(0)\}} + C(K)\mathbf{1}_{\{K\geq R(0)\}}}{\sqrt{K^3}\left(2\sqrt{K} - \sqrt{r_i} - \frac{K}{\sqrt{r_{i+1}}}\right)} dK \; .
\end{aligned}$$

- Evaluate $IIL^{v3}$ with the same risk-neutral density we constructed for $IIL^{v2}$ (Papanicolaou et al. (2025)).

# Implied Impermanent Loss ($IIL^{v3}$) – II

Given the ranges $[S_\ell(t), S_u(t)] = [(1-\alpha)R(t), (1+\alpha)R(t)]$ for $\alpha \in (0,1)$, we estimate both IIL and RIL.

- IIL
  - For v3, integrate the $IL$ across strikes $K \in [S_\ell(t), S_u(t)]$

$$IIL^{v3} = \frac{1}{T/365} \int_{S_\ell(t)}^{S_u(t)} IL(K)\, dK \ . \tag{4}$$

  - For v2, we follow Papanicolaou et al. (2025)

$$IIL^{v2} = \widetilde{\mathbb{E}}^{Q}\left[\mathrm{IL}\,(T)\right] = -\frac{\mathbb{E}^{Q}\left[P_2(T)\ln\left(\frac{R(T)}{R(0)}\right)\right]}{4e^{rT}P_2(0)}.$$

- RIL
  - For v3, instantaneous IL (Heimbach et al. (2022)) is computed hourly using the same ranges:

$$RIL_t^{v3} = IL\Big(R(t-\Delta t), R(t), S_\ell(t), S_u(t), \tilde{L}\Big)\,.$$

  - RIL is then aggregated over a rolling window of $T \cdot 24$ hours and annualized.
  - For v2, RIL is given by $1/8$ times the annualized variance of the relative price.
- ILRP (impermanent loss risk premium), for a given pair and range, is defined as

$$ILRP := IIL - RIL.$$

# Data I: Options and Implied IL Measures

- Options data: implied volatility surfaces from Amberdata, based on Deribit (the largest centralized crypto options trading exchange).
  - Tokens: BTC, ETH, SOL, XRP, BNB.
  - Maturities: 7 and 30 days.
  - Five delta-based moneyness points for calls and puts.
  - Coverage: BTC and ETH from April 2019; SOL and XRP from March 2024; BNB from October 2024.
- We recover risk-neutral densities from option prices using Breeden and Litzenberger (1978):

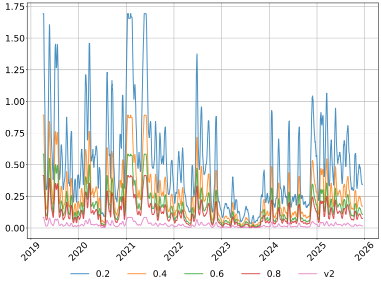
$$\left.\frac{\partial^2 C}{\partial K^2}\right|_{K=x} = e^{-r\tau} f^Q(x). \tag{5}$$

- Estimated quantities IIL, RIL, ILRP for v2 and v3, across pools, maturities, and ranges.
- Pools are grouped into:
  - Vola-Vola: neither token is a stablecoin (BTC-ETH, ETH-SOL, ...).
  - Vola-Stable: volatile token and a stablecoin (BTC-USD, ETH-USD, ...).

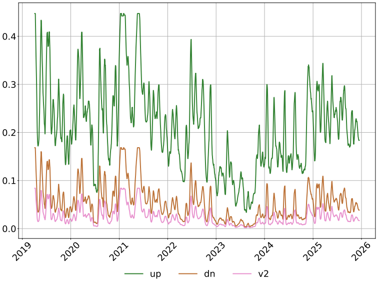
# Data II: Regression Design

- **Data sources: (The Graph)**
  - Uniswap v3 pool data: prices, TVL, volume, fees, liquidity.
  - LP transaction data.
- **LP actions:**
  - Mint: open/add liquidity.
  - Burn: remove liquidity.
  - Each event includes LP, time, range, liquidity, and token amounts.
- **Panel construction:**
  - LP $\times$ pool panel with daily unit of observation.
- **Dependent variables:**
  - *Switch*: for active rebalancing (LP changes price range).
  - *Switch Widened*: conditional on switching, the new range is wider.
  - *Liquidity ranges* $\alpha_{i,t}$.
- **Explanatory variables:**
  - *IIL*: expected impermanent loss (risk).
  - *ILRP*: compensation for bearing that risk.
- **Controls:**
  - Pool conditions: log(Price), log(TVL), log(Volume).

# Implied Impermanent Loss ($IIL^{v3}$) – I




(a) $IIL$ for $\alpha = 0.2, 0.4, 0.6, 0.8$ and v2




(b) IIL for $\alpha =$ up, down, and v2

Figure: **IIL for BTC-ETH across Ranges.** The figure reports the $IIL^{v3}$ for the BTC-ETH pool at a 7-day maturity. The left column shows $\alpha = 0.2, 0.4, 0.6, 0.8$ and v2, while the right column compares the up, down, and v2 ranges.

- IIL increases when liquidity is concentrated: a narrower range increases leverage.
- Large IIL up: When prices go up, LPs gradually sell the rising token, and opportunity costs become larger.

# Implied Impermanent Loss ($IIL^{v3}$) – II

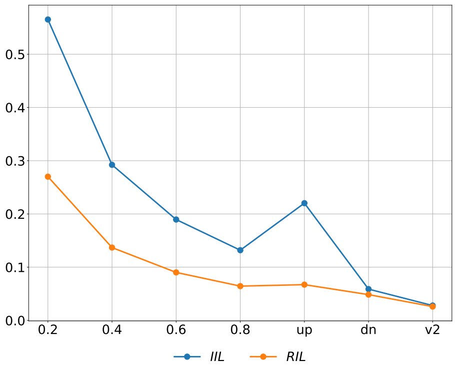
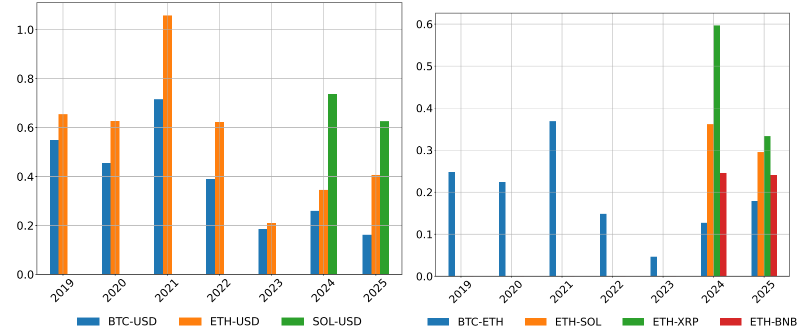



Figure: **IIL and RIL for BTC-ETH pool across Ranges.** The figure reports the average IIL and RIL for the BTC-ETH pool across ranges at a 7-day maturity.

- IIL is higher when liquidity is concentrated.
- For v3, IIL is above RIL; for v2, the difference is zero (Papanicolaou et al. (2025)).

# Implied Impermanent Loss ($IIL^{v3}$) – III




(a) $IIL^{0.6}$: Vola-Stable

(b) $IIL^{0.6}$: Vola-Vola

Figure: **IIL for Vola-Stable and Vola-Vola Pools, 60% Range.** The figure reports the average for the $IIL^{0.6}$ for BTC and ETH pools at a 7-day maturity.

- Exotic pools (e.g., ETH-XRP, ETH-BNB, and ETH-BNB) show higher IIL compared to BTC-ETH.
- Vola-Stable pools display a larger IIL than Vola-Vola pools.
- IIL is higher for Vola-Stable pools because in Vola-Vola pools, individual tokens move together and cause less impermanent loss.

# Implied Impermanent Loss ($IIL^{v3}$) – IV

| | | BTC–ETH | | BTC–USD | |
|---|---|---|---|---|---|
| Series | $\alpha$ | Mean | $Q^{0.1\%}$ | Mean | $Q^{0.1\%}$ |
| ILRP | 0.4 | 0.177 | -0.020 | 0.307 | -2.689 |
| ILRP | 0.6 | 0.120 | -0.013 | 0.199 | -1.715 |
| ILRP | 0.8 | 0.086 | -0.010 | 0.139 | -1.224 |
| ILRP | up | 0.063 | -0.009 | 0.252 | -0.623 |
| ILRP | dn | 0.005 | -0.057 | 0.037 | -0.846 |
| ILRP | v2 | 0.004 | -0.026 | 0.003 | -0.870 |

Table: **Mean and $Q^{0.1\%}$ ILRP values for BTC-ETH and BTC-USD (7 days) across Ranges.**

- Higher average ILRP when liquidity is more concentrated.
- ILRP gets negative in market turmoils (similar to VRP for equities).

## LP Behavior I

- How do risk (IIL) and risk compensation (ILRP) affect LPs' liquidity provision and rebalancing behavior?
- To answer, we estimate the following equation

$$\Delta Y_{i,t\to t+1} \;=\; \beta\,\mathrm{Risk}_{i,t} \;+\; \boldsymbol{\gamma}'\,\mathbf{X}_{i,t} \;+\; \mu_i \;+\; \lambda_{y(t)} \;+\; \varepsilon_{i,t}, \tag{6}$$

- Outcome variable (1-step ahead):
  - *Switch*.
  - *Switch Widened*.
  - $\alpha_{i,t}$ .
- Measures: IIL (risk); ILRP (risk compensation).
- Specification: Entity fixed effects and year fixed effects (annual FE). Standard errors clustered at the entity level.

# LP Behavior II

| | (1) Switch | (2) Switch | (3) Switch widened | (4) Switch widened |
|---|---|---|---|---|
| $\mathrm{IIL}_t$ | 0.025*** | | 0.038*** | |
| | (0.002) | | (0.001) | |
| $\mathrm{ILRP}_t$ | | -0.003*** | | -0.002*** |
| | | (0.001) | | (0.000) |
| $\log(\mathrm{Price})_t$ | -0.005*** | -0.005*** | -0.002*** | -0.002*** |
| | (0.002) | (0.002) | (0.001) | (0.001) |
| $\log(\mathrm{TVL})_t$ | 0.006*** | 0.005*** | 0.004*** | 0.003*** |
| | (0.002) | (0.002) | (0.001) | (0.001) |
| $\log(\mathrm{Volume})_t$ | -0.001 | 0.000 | -0.001* | 0.001 |
| | (0.001) | (0.001) | (0.001) | (0.001) |
| Observations | 456844 | 456844 | 456844 | 456844 |
| $R^2$ (within) | 0.001 | 0.000 | 0.003 | 0.000 |
| Entity FE | Yes | Yes | Yes | Yes |
| Year FE | Yes | Yes | Yes | Yes |
| SE | Clustered (entity) | Clustered (entity) | Clustered (entity) | Clustered (entity) |

Table: Panel regressions of forward-looking LP behavior on *IIL* and *ILRP*. (1)–(2) consider a switching indicator; (3)–(4) consider a conditional indicator on switching. The table includes pool–LP and time fixed effects, controls, and clustered SEs. Variables are standardized to have a mean of zero and a variance of 1.

- Higher IIL: greater probability of switching and widening the range (managing risk).
- Higher ILRP: lower incentive to widen the range (higher risk compensation).

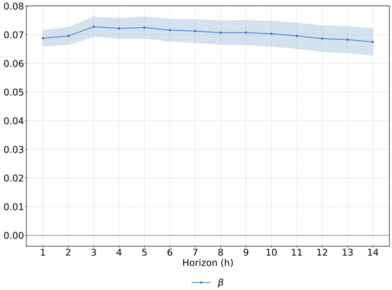
# LP Behavior III

| | (1) $\Delta\alpha$ | (2) $\Delta\alpha$ |
|---|---|---|
| $\mathrm{IIL}_t$ | 0.087*** | |
| | (0.002) | |
| $\mathrm{ILRP}_t$ | | -0.005*** |
| | | (0.000) |
| $\log(\mathrm{Price})_t$ | 0.002*** | 0.003*** |
| | (0.001) | (0.001) |
| $\log(\mathrm{TVL})_t$ | -0.002*** | -0.006*** |
| | (0.001) | (0.001) |
| $\log(\mathrm{Volume})_t$ | 0.000 | 0.011*** |
| | (0.001) | (0.001) |
| Observations | 456675 | 456675 |
| $R^2$ (within) | 0.097 | 0.003 |
| Entity FE | Yes | Yes |
| Year FE | Yes | Yes |
| SE | Clustered (entity) | Clustered (entity) |

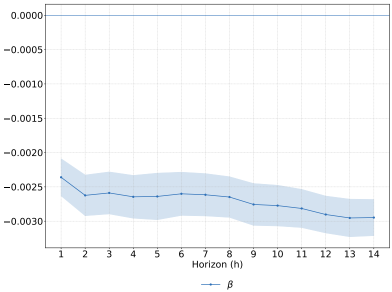
Table: **Forward changes in LP outcomes, *IIL* and *ILRP*.** The table reports panel regressions of one-period-ahead changes in $\alpha$ on pool-level IIL and ILRP. All specifications include pool–LP and year fixed effects, control variables, and standard errors clustered at the pool–LP level. Variables are standardized to have a mean of zero and a variance of 1.

- Higher $IIL_t$: LPs reallocate liquidity from narrow ranges to wider ranges.
- Higher $ILRP_t$: reallocation to tighter bands.

# Robustness




(a) $\Delta\alpha$: *IIL*




(b) $\Delta\alpha$: *ILRP*

Figure: Response of $\Delta\alpha$ to *IIL* and *ILRP*. Specifications include pool–LP and year fixed effects, control variables, and standard errors clustered at the pool–LP level. Each panel displays the estimated coefficient $\beta$ and its 95% confidence interval.

- ILL: positive and stable effect on $\Delta\alpha$ across all horizons $\rightarrow$ robust and persistent response.
- ILRP: negative effect that becomes slightly stronger over time $\rightarrow$ weaker but systematic impact.

# Conclusion

- Mathematical Framework and Estimation:
  - Extend Risk-Neutral ("Implied") Impermanent Loss (IIL) for v3.
  - Density via HJ bounds (no index volatility available).
- Empirically
  - Computed IIL for v3 from traded derivatives across 7 pools and multiple ranges.
  - IIL is higher for concentrated ranges.
  - Higher risk in exotic pools.
  - Increases in IIL prompt widening of ranges.
  - Increases in the ILRP encourage re-centering of liquidity toward the market price.

$\Rightarrow$ **Many open questions to be answered!**

$$t^h a_n (k) \;\; y_o [u] \; !$$

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# References I


Alexander, C. and Imeraj, A. (2021). The bitcoin vix and its variance risk premium. *The Journal of Alternative Investments*, 23(4):84–109.

Bakshi, G., Kapadia, N., and Madan, D. (2015). Stock return characteristics, skew laws, and the differential pricing of individual equity options. *The Review of Financial Studies*, 16(1):101–143.

Bollerslev, T., Marrone, J., Xu, L., and Zhou, H. (2014). Stock return predictability and variance risk premia: Statistical inference and international evidence. *Journal of Financial and Quantitative Analysis*, 49(03):633–661.

Bollerslev, T., Tauchen, G., and Zhou, H. (2009). Expected stock returns and variance risk premia. *The Review of Financial Studies*, 22(11):4463–4492.

Breeden, D. T. and Litzenberger, R. H. (1978). Prices of state-contingent claims implicit in option prices. *The Journal of Business*, 51(4):621–651.

Carr, P. and Madan, D. (1998). *Volatility: New Estimation Techniques for Pricing Derivatives*, chapter 29 Towards a theory of volatility trading. Risk Books.

Carr, P. and Wu, L. (2009). Variance risk premiums. *The Review of Financial Studies*, 22(3):1311–1341.

Cartea, A., Drissi, F., and Monga, M. (2024). Decentralised finance and automated market making: Predictable loss and optimal liquidity provision. *arXiv Manuscript*.

Clark, J. (2020). The replicating portfolio of a constant product market. *SSRN Manuscript*.

Clark, J. (2021). The replicating portfolio of a constant product market with bounded liquidity. *SSRN Manuscript*.

Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999). A guide to volatility and variance swaps. *The Journal of Derivatives*, 6(4):9–32.

Drechsler, I. and Yaron, A. (2011). What's vol got to do with it. *The Review of Financial Studies*, 24(1):1–45.

# References II


Feunou, B., Jahan-Parvar, M. R., and Okou, C. (2018). Downside variance risk premium. *Journal of Financial Econometrics*, 16(3):341–383.

Fukasawa, M., Maire, B., and Wunsch, M. (2023). Weighted variance swaps hedge against impermanent loss. *Quantitative Finance*, 23(6):901–911.

Hansen, L. P. and Jagannathan, R. (1991). Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99(2):225–262.

Heimbach, L., Schertenleib, E., and Wattenhofer, R. (2022). Risks and returns of uniswap V3 liquidity providers. In *4th ACM Conference on Advances in Financial Technologies*, Cambridge, Massachusetts, U.S.A.

Kilic, M. and Shaliastovich, I. (2019). Good and bad variance premia and expected returns. *Management Science*, 65(6):2445–2945.

Lee, R. (2010a). *Encyclopedia of Quantitative Finance*, chapter Corridor Variance Swap. John Wiley & Sons.

Lee, R. (2010b). *Encyclopedia of Quantitative Finance*, chapter Weighted variance swap. John Wiley & Sons.

Lehar, A. and Parlour, C. (2023). Decentralized exchange: The uniswap automated market maker. *SSRN Manuscript*.

Li, T. N., Naik, S., Papanicolaou, A., and Schönleber, L. (2024). Yield farming for liquidity provision. *SSRN Manuscript*.

Milionis, J., Moallemi, C. C., Roughgarden, T., and Zhang, A. L. (2024). Automated market making and loss-versus-rebalancing. *arXiv Manuscript*.

Papanicolaou, A., Schönleber, L., and Li, T. N. (2025). The implied impermanent loss in decentralized liquidity provision. *SSRN Manuscript*.

# Appendix

# Mathematical Framework

# Continuous-Time Approach – Single Pool – AMM

- The constant product rule for a LP ($N_1(t)$ and $N_2(t)$ amount of tokens):

$$L := \sqrt{N_1(t)N_2(t)}. \tag{7}$$

- From this constant product rule (details) (7), a relative price emerges:

$$R(t) = \frac{N_2(t)}{N_1(t)} = \text{token 2 per token 1}, \tag{8}$$

- In terms of absolute price, we assume

$$R(t) = \frac{P_1(t)}{P_2(t)} = \frac{\text{dollar per token 1}}{\text{dollar per token 2}} = \text{token 2 per token 1}, \tag{9}$$

where $P_i(t)$ for $i = 1$ and 2 are the exogenous prices of tokens.

# Single Pool – Token Dynamics

- The prices of tokens in the LP follow GBMs (real-world dynamics)

$$dP_i(t) = \mu_i P_i(t)dt + \sigma_i P_i(t)dB_i(t), \quad i = 1,\ 2, \tag{10}$$

  where $B_i(t)$ for $i = 1$ and 2 are two correlated Brownian motions.
- Apply Itô to get the SDE for the relative price $R(t) = P_1(t)/P_2(t)$ and:

$$dR(t) = d\left(\frac{P_1(t)}{P_2(t)}\right) = \cdots = \mu_R(t)R(t)dt + \sigma_R(t)R(t)dB_R(t),$$

$$dN_1(t) = d\left(\frac{L}{\sqrt{R(T)}}\right), \tag{11}$$

$$dN_2(t) = d(L\sqrt{R(T)}) \tag{12}$$

  where $\sigma_R^2 = \sigma_1^2(t) - 2\rho\sigma_1(t)\sigma_2(t) + \sigma_2^2(t)$,
  $\mu_R\left(t\right) = \mu_1 - \mu_2 + \sigma_2^2\left(t\right) - \rho\sigma_1\left(t\right)\sigma_2\left(t\right)$.

# Single Pool – Impermanent Loss

- ... which allows us to derive the expression for the impermanent loss

$$dIL(t) := \frac{\mathcal{V}^{\text{held}}_{t+dt} - \mathcal{V}^{\text{staked}}_{t+dt}}{\mathcal{V}^{\text{held}}_{t+dt}} = \cdots = \frac{\sigma_R^2(t)}{8} dt \leq 0, \quad \forall t \geq 0, \tag{13}$$

- impermanent loss $\approx$ realized volatility of the relative price ($R$)
- $\sigma_R^2 = \sigma_1^2(t) - 2\rho\sigma_1(t)\sigma_2(t) + \sigma_2^2(t)$
- $dIL(t)$ more negative for high variances and high negative correlation
- $dIL(t) \approx 0$ if $\sigma_1^2 = \sigma_2^2(t) \approx 0$, or $\sigma_1^2(t) = \sigma_2^2(t)$, $\rho = 1$
- details
- See Li et al. (2024) for a rigorous derivation.

# Impermanent Loss

# Impermanent Loss (IL)

- IL: Opportunity-cost dynamics between providing liquidity and holding the underlying tokens to potentially profit from the price movement.
- IL: prices of tokens diverge (no matter in which direction), causing LPs to underperform a basic buy-and-hold strategy.
    - IL $\neq$ LPs experiences a negative return:
      asset value from buy-and-hold $\geq$ asset value from the liquidity provision.

# Impermanent Loss (IL)

$$dI_t := \frac{(N_1(t)dP_1(t) + N_2(t)dP_2(t)) - d\left(N_1(t)P_1(t) + N_2(t)P_2(t)\right)}{N_1(t)P_1(t) + N_2(t)P_2(t)} \tag{14}$$

$$= \frac{\sigma_R^2}{8} dt \leq 0, \quad \forall t \geq 0, \tag{15}$$

# Implied Impermanent Loss

# Change of Measure – I

Recall our pair of SDEs for token prices,

$$dP_1(t) = rP_1(t)dt + \sigma_1(t)P_1(t)dB_1(t)$$
$$dP_2(t) = rP_2(t)dt + \sigma_2(t)P_2(t)dB_2(t) \ ,$$

where $B_1(t)$ and $B_2(t)$ are risk-neutral standard Brownian motions with correlation parameter $\rho$. Here, we let $\mathbb{Q}$ denote the risk-neutral measure.

# Change of Measure – II

$$\begin{aligned}
\text{Call}^{spr}(T,K) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}(P_1(T) - KP_2(T))^+ && (16)\\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(P_2(T)(R(T)-K)^+\right) && (17)\\
&= P_2(0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P_2(T)}{P_2(0)e^{rT}}(R(T)-K)^+\right) && (18)\\
&= P_2(0)\widetilde{\mathbb{E}}^{\mathbb{Q}}(R(T)-K)^+ \ , && (19)
\end{aligned}$$

where $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_T = \frac{P_2(T)}{P_2(0)e^{rT}}$, under which $R(T)$ is a $\widetilde{\mathbb{Q}}$ martingale.

$$\begin{aligned}
\widetilde{\mathbb{E}}^{\mathbb{Q}}[\frac{P_1(T)}{P_2(T)}] &= \mathbb{E}^{\mathbb{Q}}[\frac{P_2(T)}{P_2(0)e^{rT}}\frac{P_1(T)}{P_2(T)}] = \mathbb{E}^{\mathbb{Q}}[\frac{P_1(T)}{P_2(0)e^{rT}}] = \frac{\mathbb{E}^{\mathbb{Q}}[P_1(T)]}{P_2(0)e^{rT}} && (20)\\
&= \frac{P_1(0)e^{rT}}{P_2(0)e^{rT}} = \frac{P_1(0)}{P_2(0)} && (21)
\end{aligned}$$

# Change of Measure – Girsanov – I

Let

$$X(t) = \int_0^t \sigma_2(s) dB_2^{\mathbb{Q}}(s),$$

which is a martingale under $\mathbb{Q}$. The Dolean-Dade exponent is

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_T = \mathcal{E}(X(T)) = e^{X(T) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt} = \frac{P_2(T)}{e^{rT} P_2(0)} \ .$$

Girsanov Theorem: Given $W(t)$ that is Brownian motion under $\mathbb{Q}$, we define

$$\widetilde{W}^{\mathbb{Q}}(t) := W^{\mathbb{Q}}(t) - [W^{\mathbb{Q}}, X](t) \ ,$$

which is Brownian motion under $\widetilde{\mathbb{Q}}$.

- Hence, $\widetilde{B}_2^{\mathbb{Q}}(t) = B_2^{\mathbb{Q}}(t) - \int_0^t \sigma_2(s) ds$ is $\widetilde{\mathbb{Q}}$ Brownian motion.

# Change of Measure – Girsanov – II

Suppose for $\mathbb{Q}$ Brownian motion $W(t)$ that $dB_2^{\mathbb{Q}}(t)dW(t) = \rho dt$. Then by Girsanov theorem, define

$$\widetilde{W}^{\mathbb{Q}}(t) = W^{\mathbb{Q}}(t) - [W, X](t) = W^{\mathbb{Q}}(t) - \rho \int_0^t \sigma_2(s)ds \ ,$$

which is $\widetilde{\mathbb{Q}}$ Brownian motion. In particular, for the token prices $P_1$ and $P_2$, under the new measure we have

$$\begin{aligned} dP_1(T) &= rP_1(T)dt + \sigma_1(t)P_1(T)\left(dB_1^{\mathbb{Q}}(t) - \rho\sigma_2(t)dt + \rho\sigma_2(t)dt\right) \\ &= \left(r + \rho\sigma_1(t)\sigma_2(t)\right)P_1(T)dt + \sigma_1(t)P_1(T)d\widetilde{B}_1^{\mathbb{Q}}(t) \\ dP_2(T) &= \left(r + \sigma_2^2(t)\right)P_2(T)dt + \sigma_2(t)P_2(T)d\widetilde{B}_2^{\mathbb{Q}}(t) \end{aligned}$$

where $\widetilde{B}_1^{\mathbb{Q}}(t) = B_1^{\mathbb{Q}}(t) - \rho\int_0^t \sigma_2(s)ds$. Notice that $e^{-rt}/P_2(T)$ is a $\widetilde{\mathbb{Q}}$ martingale (i.e., the discounted price of the original 'currency' is a martingale under the change of numeraire).

# Implied Impermanent Loss – Details

Remember: Volatility Swap (Demeterfi et al. (1999); Carr and Madan (1998); Bakshi et al. (2015)):

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \sigma_i^2(t)dt\right] = \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{dP_i(t)}{P_i(t)}\right] - \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\log\left(\frac{P_i(T)}{P_i(0)}\right) \quad \text{for } i = 1, 2. \tag{22}$$

Then for the log-contracts (for $P_1$ and $P_2$ under tilde measure and $r = 0$),

$$\frac{2}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_1(T)/P_1(0)) = \frac{2}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\int_0^T\left(\rho\sigma_1(t)\sigma_2(t) - \frac{1}{2}\sigma_1(t)^2\right)dt$$

$$\frac{2}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_2(T)/P_2(0)) = \frac{1}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\int_0^T \sigma_2(t)^2 dt\ .$$

## Implied Impermanent Loss – III

Which we put together to obtain the $\widetilde{\mathbb{Q}}$-valuation of the IL,

$$\widetilde{\mathbb{E}}^{\mathbb{Q}}\mathsf{IL}_T = -\frac{1}{8T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\int_0^T \Big(\sigma_1(t)^2 + \sigma_2(t)^2 - 2\rho\sigma_1(t)\sigma_2(t)\Big)\,dt \tag{23}$$

$$= -\frac{1}{4T}\left(\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_2(T)/P_2(0)) - \widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_1(T)/P_1(0))\right) \tag{24}$$

$$= \frac{1}{4T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(R(T)/R(0)), \tag{25}$$

where we took advantage of the logarithm property

$$\log\left(\frac{R_T}{R_0}\right) = \log\left(\frac{P_1(T)}{P_1(0)}\right) - \log\left(\frac{P_2(T)}{P_2(0)}\right). \tag{26}$$

# Implied Impermanent Loss – v3

- We have that $dR(t) = \mu_R R(t)dt + \sigma_R(t)R(t)dW_R(t)$. And,

$$
\begin{aligned}
&R_{t+\Delta t}\mathbf{1}_{R_{t+\Delta t}<R_l} - R_{t+\Delta t}\mathbf{1}_{R_t<R_l} \\
&\approx (R_t + dR_t)(\mathbf{1}_{R_t<R_l} - \delta_{R_l}(R_t)dR_t - \frac{1}{2}\delta'_{R_l}(R_t)(dR_t)^2) - (R_t + dR_t)\mathbf{1}_{R_t<R_l} \\
&= -\delta_{R_l}(R_t)(R_t dR_t + (dR_t)^2) - \frac{1}{2}\delta'_{R_l}(R_t)R_t(dR_t)^2
\end{aligned}
$$

- Then,

$$
\begin{aligned}
&\mathbf{1}_{R_{t+\Delta t}>R_u} - \mathbf{1}_{R_t>R_u} \\
&\approx \mathbf{1}_{R_t>R_u} + \delta_{R_u}(R_t)dR_t + \frac{1}{2}\delta'_{R_u}(R_t)(dR_t)^2 - \mathbf{1}_{R_t>R_u} \\
&= \delta_{R_u}(R_t)dR_t + \frac{1}{2}\delta'_{R_u}(R_t)(dR_t)^2
\end{aligned}
$$

# Implied Impermanent Loss – v3

- Finally

$$\left(2\sqrt{R_{t+\Delta t}} - \sqrt{R_l} - \frac{R_{t+\Delta t}}{\sqrt{R_u}}\right)\mathbf{1}_{R_l \le R_{t+\Delta t} < R_u} - \left(\frac{R_t + R_{t+\Delta t}}{\sqrt{R_t}} - \sqrt{R_l} - \frac{R_{t+\Delta t}}{\sqrt{R_u}}\right)\mathbf{1}_{R_l \le R_t}$$

$$\approx \left(2\sqrt{R_t} + \frac{dR_t}{\sqrt{R_t}} - \frac{(dR_t)^2}{4R_t^{3/2}} - \sqrt{R_l} - \frac{R_t + dR_t}{\sqrt{R_u}}\right)$$

$$\times\left(\mathbf{1}_{R_l \le R_t < R_u} + (\delta_{R_l}(R_t) - \delta_{R_u}(R_t))dR_t + \frac{1}{2}(\delta'_{R_l}(R_t) - \delta'_{R_u}(R_t))(dR_t)^2\right)$$

$$-\left(\frac{2R_t + dR_t}{\sqrt{R_t}} - \sqrt{R_l} - \frac{R_t + dR_t}{\sqrt{R_u}}\right)\mathbf{1}_{R_l \le R_t < R_u}$$

## Implied Impermanent Loss – v3

- The LVR in the band is, after a little bit of algebra

$$dLVR_t \frac{\sigma_R^2(t)}{4} \sqrt{R_t} \mathbf{1}_{R_l \le R_t < R_u} dt \ .$$

- The IL is

$$dIL_t = \frac{\sigma_R^2(t)}{4\left(2\sqrt{R_t} - \sqrt{R_l} - \frac{R_t}{\sqrt{R_u}}\right)} \sqrt{R_t} \mathbf{1}_{R_l \le R_t < R_u} dt \ .$$

## Example of Impermanent Loss ($IL^{v3}$) - price range

Let's consider this example, where

- Assume: Token A, $S_0 = 100$ USD (USDT as the stablecoin),
- Range for the LP to supply liquidity: $[S_l, S_u] = [81, 121]$ USD,
- Let's define $\quad p = \sqrt{S} = 10, \quad a = \sqrt{S_l} = 9, \quad b = \sqrt{S_u} = 11,$
- At time $t_0$: the LP owns $x_0 = 100$ Token A worth 10,000 USD at the price $S_0 = 100$,
- With `liquidity` $L$ and `price` $S \in [S_l, S_u]$ the token balances are

$$\text{Token A} = x(S) = L\frac{b-p}{pb} \quad \text{USDT} = y(S) = L(p-a)\ . \tag{27}$$

- Imposing that the position contains exactly $x_0 = 100$ Token A at $S_0$ implies $L = 11,000$. Hence, the corresponding USDT position at $S_0$ is

$$y_0 = L(p-a) = 11,000 \times (10-9) = 11,000 \text{ USDT}. \tag{28}$$

- Time $t_0$: initial portfolio value $V_0 = x_0 S_0 + y_0 = 100 \times 100 + 11,000 = 21,000$ USD.
- Time $t_1$: the market price of Token A increases from 100 USD to 121 USD.

**Holding**

- 100 Token A $\times\, 121 = 12,100$ USD
- $+$ initially held $11,000$ USDT

**LP**

- $S \to S_u$, we have a position only in USDT.
- $y_{\text{pos}}(S_u) = 11,000 \times (11-9) = 22,000$ USDT.

## Implied Impermanent Loss ($IIL^{v3}$) – III

- Let $P_{1,t}$ and $P_{2,t}$ denote the prices of tokens 1 and 2 at time $t$, and define the price ratio $S_t = P_{1,t}/P_{2,t}$.
- For a liquidity range $[S_\ell, S_u]$ and normalized liquidity $\tilde{L} = 1$, we discretize the admissible price support using $n$ grid points, $K \in [S_\ell, S_u]$,
- and compute IL payoffs for deviations of the future price ratio $R$ from each grid point

$$\Pi(K, S) = \mathbf{1}_{\{K<S_0\}} \max(K - S, 0) + \mathbf{1}_{\{K \geq S_0\}} \max(S - K, 0) \ . \tag{29}$$

- Let $\tilde{q}(R)$ denote the estimated pricing kernel (or state-price density) of the future price ratio.
- For each grid point $K$, the impermanent loss price is computed as

$$IL(K) = \frac{\int \Pi(K, S)\, \tilde{q}(S)\, dS}{K^{3/2}\left(2\sqrt{K} - \sqrt{S_\ell} - \frac{K}{\sqrt{S_u}}\right)} \ , \tag{30}$$

- and we define the implied impermanent loss for v3 as

$$IIL^{v3} = \frac{1}{T/365} \int_{S_\ell}^{S_u} IL(K)\, dK \ . \tag{31}$$

# Implied Impermanent Loss ($IIL^{v2}$ and $IIL^{v3}$)

We consider a range $[S_{\ell,t}, S_{u,t}]$ using $\alpha$ as range

- For $\alpha \in (0,1)$, we consider symmetric ranges

$$S_{\ell,t} = (1-\alpha)S_{t-1} \ , \qquad S_{u,t} = (1+\alpha)S_{t-1} \ . \tag{32}$$

- An upward-only range (up) with $S_{\ell,t} = S_{t-1}(1-0.01)$ and $S_{u,t} = (1+10)S_{t-1}$.
- A downward-only range (dn) with $S_{\ell,t} = S_{t-1}10^{-4}$ and $S_{u,t} = S_{t-1}(1+0.01)$.
- For the IIL of v2, we follow **?**, calculating the following equation

$$\mathrm{IIL}^{v2} = \widetilde{\mathbb{E}}^{Q}\left[\mathrm{IL}\left(T\right)\right] = -\frac{\mathbb{E}^{Q}\left[P_2\left(T\right)\ln\left(\frac{R(T)}{R(0)}\right)\right]}{4e^{rT}P_2\left(0\right)} \ . \tag{33}$$

- The resulting time series $IIL_t^{v3,\alpha}(T)$ is computed for each pool, maturity, and liquidity concentration parameter $\alpha$.

# Realized Impermanent Loss (RIL v2 and v3)

- The instantaneous IL is computed as

$$IL_t = IL\Big(R_{t-1}, R_t, S_{\ell,t}, S_{u,t}\ , \tilde{L}\Big)\ , \tag{34}$$

  where $IL(\cdot)$ denotes the Uniswap v3 IL function as outlined in Heimbach et al. (2022).
- We define realized impermanent loss at time $t$ as

$$RIL_t^{v3} = -IL_t\ .$$

- For each maturity $T$, the RIL is aggregated over a rolling window of $T \cdot 24$ hourly observations using a rolling mean,

$$RIL_t^{v3}(T) = (24 \cdot 365) \cdot \frac{1}{(T \cdot 24)} \sum_{k=0}^{T \cdot 24 - 1} RIL_{t-k}^{v3}\ , \tag{35}$$

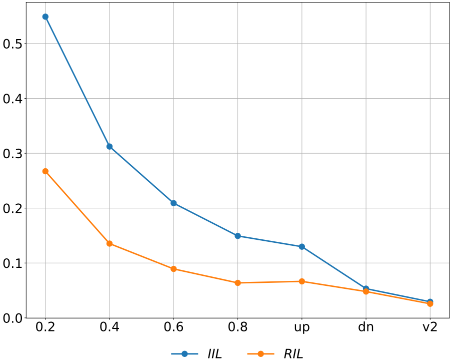
  where the window is evaluated using at least $T \cdot 24/2$ available observations.
- The RIL for v2 is simply $1/8$ times the annualized variance of the relative price and is calculated following

$$RIL_t(T) = \frac{1}{8} \cdot (24 \cdot 365) \cdot \frac{1}{(T \cdot 24)}, \left( \sum_{k=0}^{T \cdot 24 - 1} \left( \Delta \log \left( \frac{P_{1,t-k}}{P_{2,t-k}} \right)^{(i,j)} \right)^2 \right)\ .$$

- The resulting time series $RIL_t^{v3,\alpha}(T)$ is computed for each pool, maturity, and liquidity concentration parameter $\alpha$.
- The impermanent loss risk premia (ILRP) is defined as ILRP := IIL − RIL.

# Implied Impermanent Loss ($IIL^{v3}$) – I

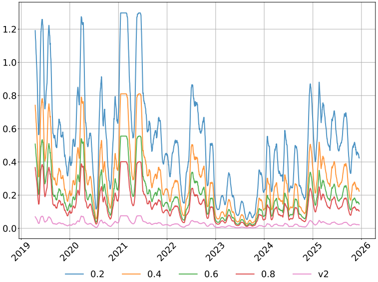
[label=Examplev3]



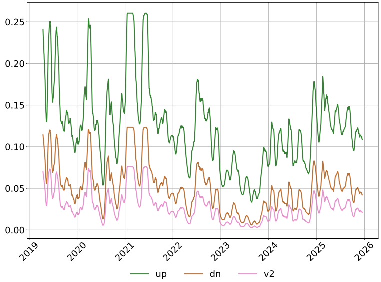

(a) Average BTC-ETH

Figure: **Average IIL for the BTC-ETH pool across Ranges - 30 days**.

# Implied Impermanent Loss ($IIL^{v3}$) – II



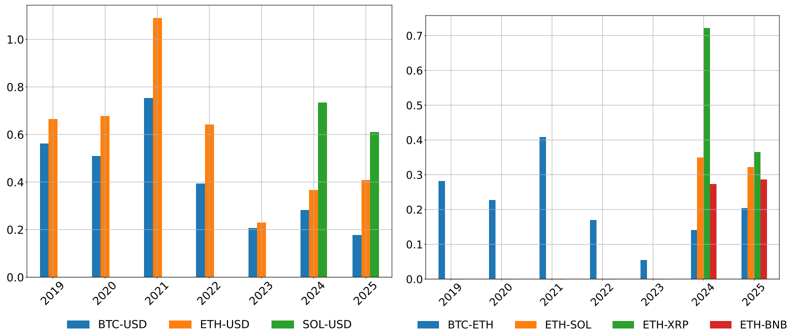

(a) *IIL* for $\alpha = 0.2, 0.4, 0.6, 0.8$ and v2




(b) IIL for $\alpha =$ up, down, and v2

Figure: **IIL for BTC-ETH across Ranges.** The figure reports the implied impermanent loss $IIL^{v3}$ for the BTC-ETH pool at a 30-day maturity. The left column shows $\alpha = 0.2, 0.4, 0.6, 0.8$ and the v2 benchmark, while the right column compares the up, down, and v2 ranges. The data are sampled daily over April 2019–December 2025 and smoothed using a 14-day rolling mean.

# Implied Impermanent Loss ($IIL^{v3}$) – III




(a) $IIL^{0.6}$: Vola-Stable (b) $IIL^{0.6}$: Vola-Vola

Figure: **IIL for Vola-Stable and Vola-Vola Pools, 60% Range.** The figure reports the average for the implied impermanent loss $IIL^{0.6}$ for the BTC and ETH pools at a 30-day maturity.

# Implied Impermanent Loss ($IIL^{v3}$) – IV

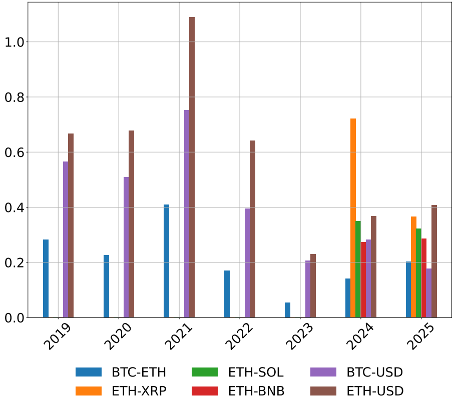
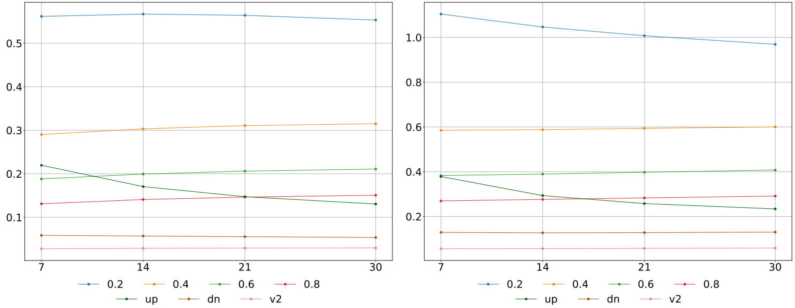



Figure: **IIL: Vola–Vola and Vola–Stable (60% Range).** The figure reports the average for the implied impermanent loss $IIL^{0.6}$ for the analysed pools at a 7-day maturity.

- Vola–Stable pools usually show higher IIL.
- IIL is higher for Vola-Stable pools because in Vola-Vola pools, individual tokens move together and cause less impermanent loss

# Implied Impermanent Loss ($IIL^{v3}$) – V




(a) Term structure - BTC-ETH

(b) Term structure - BTC-USD

Figure: **Term structure of** ***IIL***. This figure shows the term structure of the *IIL* for the BTC-ETH pool (left panel) and for BTC-USD (right panel).

- BTC-ETH: upward sloping term structure.
- BTC-USD: flat term structure.
- When liquidity is tight (or tilted upward), most of the IL risk is about near-term price moves. That's why the IIL is larger at 7 days for $\alpha = 0.2$, "up".

# Implied Impermanent Loss ($IIL^{v3}$) – VI

| | | BTC–ETH | | BTC–USD | |
|---|---|---|---|---|---|
| Series | $\alpha$ | Mean | $Q^{0.1\%}$ | Mean | $Q^{0.1\%}$ |
| ILRP | 0.2 | 0.282 | -0.055 | 0.419 | -2.863 |
| ILRP | 0.4 | 0.177 | -0.020 | 0.322 | -1.131 |
| ILRP | 0.6 | 0.120 | -0.013 | 0.224 | -0.690 |
| ILRP | 0.8 | 0.086 | -0.010 | 0.160 | -0.488 |
| ILRP | up | 0.063 | -0.009 | 0.107 | -0.366 |
| ILRP | dn | 0.005 | -0.057 | 0.038 | -0.347 |
| ILRP | v2 | 0.004 | -0.026 | 0.006 | -0.317 |

Table: **Average and $Q^{0.1\%}$ ILRP values for BTC-ETH and BTC-USD (30 days) across Ranges.**

# Optimization Details

# Joint Distribution – HJ Bounds

Hansen and Jagannathan (1991) (HJ): For any excess return $\Pi$ and a pricing kernel ($M$) we have

$$0 = \mathbb{E}^{\mathbb{P}}[\Pi M] = \mathrm{cov}(\Pi M) + \mathbb{E}[\Pi] \geq -\sigma(\Pi)\sigma(M) + \mathbb{E}[\Pi]. \tag{36}$$

HJ bound on Sharpe Ratios:

$$\sup_{\Pi:\sigma(\Pi)>0} \frac{\mathbb{E}[\Pi]}{\sigma(\Pi)} \leq \inf_{M \in \mathcal{M}} \sigma(M) \tag{37}$$

- The right-hand side motivates us to find a pricing kernel that minimizes the HJ bound.
- $M_{ij} = \frac{q_{ij}}{p_{ij}}$, where $q_{ij}$ ($p_{ij}$) is the risk-neutral (real-world) measure.
- $\mathbb{E}^{\mathbb{P}}[M] = \sum_{i,j} M_{ij} p_{ij} = 1$, and hence $\sigma^2(M) = \sum_{i,j} (M_{ij} - 1)^2 p_{ij}$.

## Joint Distribution – Optimization Problem – I

Ansatz: For a risk-neutral distribution $q$ let $\phi_{ij} := q_{ij}/\sqrt{p_{ij}}$

$$\Rightarrow \sigma^2(M) = \sum_{i,j}\left(\frac{q_{ij}}{p_{ij}} - 1\right)^2 p_{ij} = \sum_{i,j}\phi_{ij}^2 - 1 \tag{38}$$

We then formulate the following constrained optimization problem:

$$\min_{\phi:\phi_{ij}>0}\sum_{i,j}\phi_{ij}^2 \ ,$$

s.t.

$$\sum_j \phi_{i,j}\sqrt{p_{i,j}} = \mu_i$$

$$\sum_i \phi_{i,j}\sqrt{p_{i,j}} = \nu_j$$

$$\phi_{i,j} \geq 0$$

where $\mu_i, \nu_j$ represent the marginal distribution of token $(i,j)$. details

# Joint Distribution – Optimization Problem – II

Inputs:

- Risk-neutral density for the margins ($q_i$) (obtained from constructed option surfaces).
- Real-world density for the margins ($p_i$) (obtained from realized data: bivariate Gaussian estimated over a rolling window of 30 days).
- Remark: We reformulate the problem in vector/matrix to solve it. For $n = m = 100$ we obtain 200 constraints involving an $n \times m$ (10000) sparse matrix. We resort to the Matlab large sparse QP solver.

# AMM Arithmetics

# AMM Arithmetics

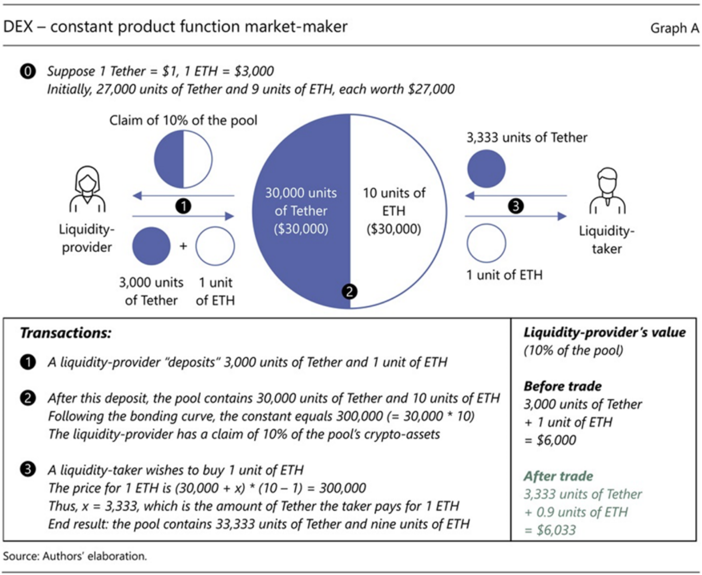



Figure: **AMM Functioning**

# Data

# Options – Deribit – I

- Deribit: so far the largest exchange for options on cryptocurrencies

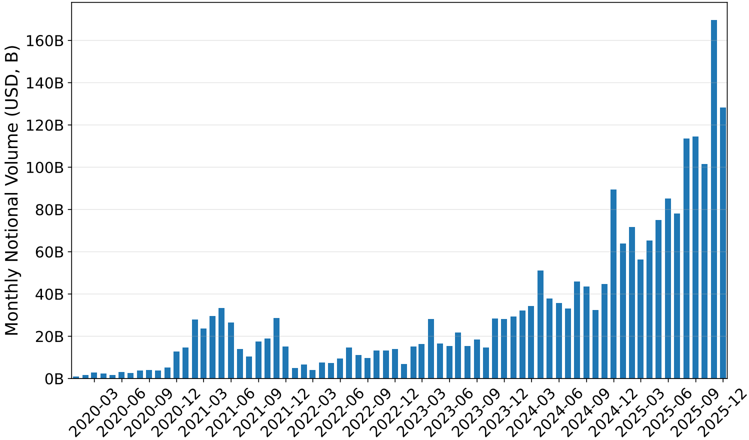



Figure: **Monthly Option Volumes on Deribit - BTC.** The figure reports the monthly total option trading volume on Deribit, measured in notional value (USD). Notional volume is defined as the total value of option contracts traded, including both on-screen trades executed directly on the exchange and block trades executed through third-party platforms. The data are sourced from Amberdata.

# Options – Deribit – II

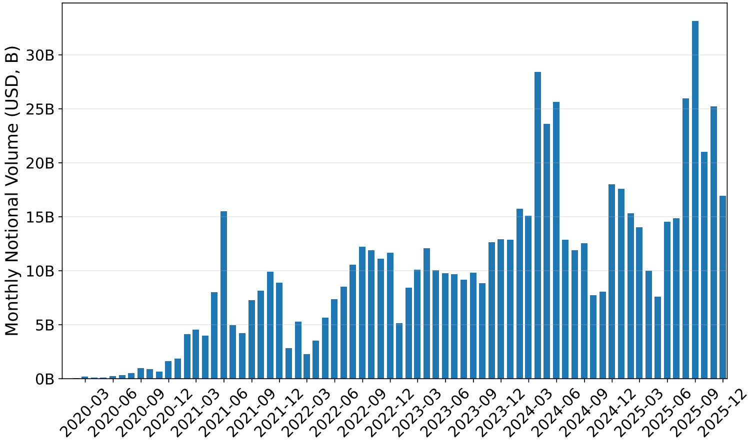



Figure: **Monthly Option Volumes on Deribit - ETH.** The figure reports the monthly total option trading volume on Deribit, measured in notional value (USD). Notional volume is defined as the total value of option contracts traded, including both on-screen trades executed directly on the exchange and block trades executed through third-party platforms. The data are sourced from Amberdata.

# DeFi and Yield Farming – Ethereum